IMPLEMENTATION OF HIGH ORDER SYMPLECTIC INTEGRATORS WITH POSITIVE STEPS IN TRACKING PROGRAMS *

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Abstract

The symplectic integrators CSABA$_\nu$ & CSBAB$_\nu$ are used in order to calculate single particles dynamics in accelerators and storage rings. These integrators present only positive steps and can be accurate up to a very high order. They are compared with already existing splitting methods of MAD-X with respect to their impact on various beam dynamics quantities, for different beam elements.

INTRODUCTION

The necessity of designing high-performance particle accelerators and storage rings led in developing a number of different tracking programs in order to propagate and study the evolution of particle trajectories. In particular, the knowledge of particle dynamics for a great interval of time (thousands or millions of turns) is essential in order to assure their long-term stability in presence of various non-linearities.

Neglecting radiating effects, the single particle motion in accelerators is represented by a composition of symplectic maps generated by the Hamiltonian of the corresponding magnetic elements. Tracking codes have to employ symplectic integrators in order to guarantee that the numerical integration preserve the Hamiltonian structure of the system, i.e. the "energy", which should be an integral of motion and prevent the particles to have an artificial "diffusion".

Preceding works proposed various symplectic integrators [1–3]. The ones that have an order of accuracy $O(\tau^\zeta)$ greater than two ($\zeta > 2$) incorporate negative "time" steps [4]. On the other hand Laskar and Robutel [5], by generalising and enriching McLachlan's work [6], proposed a set of symplectic integrators with only positive steps even in cases with $\zeta > 2$. These symplectic integrators are valid for perturbed Hamiltonians of the form $\mathcal{H} = A + \epsilon B$, where A and B are independently integrable.

CSABA$_\nu$ & CSBAB$_\nu$ INTEGRATORS

The Hamilton's equations of motion for a problem with N degrees of freedom can be written as:

$$\frac{d\vec{\mathbf{X}}}{dt} = \{\mathcal{H}, \vec{\mathbf{X}}\} = L_{\mathcal{H}}\vec{\mathbf{X}} \tag{1}$$

where t is the independent variable and $\vec{\mathbf{X}} = (q_1, p_1, ..., q_N, p_N)$ is a generalized vector containing the canonical conjugate variables q_i and p_i. The expression $\{\mathcal{H}, \vec{\mathbf{X}}\}$ refers to the Poisson brackets that are described by $\{\omega(q,p), \lambda(q,p)\} = \sum_{j=1}^{N}\left[\frac{d\omega}{dp_j}\frac{d\lambda}{dq_j} - \frac{d\omega}{dq_j}\frac{d\lambda}{dp_j}\right]$, the action of the Lie operator $L_{\mathcal{H}}$ is defined by $L_\omega\lambda = \{\omega, \lambda\}$. The distance s along the lattice, instead of the time, is used as the integration variable. The formal solution of Eq.(1) from the initial position s^i to the final one s^f with $\tau = s^f - s^i$ is given by:

$$\vec{\mathbf{X}}^f = \sum_{n\geq 0}\frac{\tau^n}{n!}L^n_{\mathcal{H}}\vec{\mathbf{X}}^i = e^{\tau L_{\mathcal{H}}}\vec{\mathbf{X}}^i \tag{2}$$

where i and f stand for the initial and final values respectively, $\vec{\mathbf{X}}(s^i) \equiv \vec{\mathbf{X}}^i$ are the initial conditions and $e^{\tau L_{\mathcal{H}}}$ is the Lie transformation.

Using the Backer-Campbell-Hausdorff (BCH) theorem, the Lie transformation $e^{\tau L_{\mathcal{H}}} = e^{\tau(L_A + L_{\epsilon B})}$ can be re-expressed by an infinite concatenation of Lie transformations $e^{c_j\tau L_A}$ and $e^{d_j\tau L_{\epsilon B}}$ ($j = 1, 2, \dots$), if A and B do not commute ($\{A, B\} \neq 0$). Truncating this infinite series up to a certain order of j, results in a symplectic integrator of an effective Hamiltonian, $\widetilde{\mathcal{H}} = \mathcal{H} + O(\tau^\zeta)$. Using symplectic integrators, the dynamical behavior of the nominal Hamiltonian ($\mathcal{H}$) is approximated by an effective one ($\widetilde{\mathcal{H}}$) introducing an error term of the order $O(\tau^\zeta)$. Increasing the order of j (the number of splittings) and choosing appropriately the values of c_j and d_j, the deviation of $\widetilde{\mathcal{H}}$ from $\mathcal{H}$ is reduced (ζ is increased). When the term $\mathcal{F} = \{\{A, B\}, B\}$ is integrable, it can be proved that the symplectic integrators CSABA$_\nu$ & CSBAB$_\nu$ can be used [5]. They are described by the following recursion relations:

$$CSABA_{2n} = e^{-uf_{2n}L_{\mathcal{F}}}e^{c_1\tau L_A}e^{d_1\tau L_{\epsilon B}}\dots e^{d_n\tau L_{\epsilon B}}e^{c_{n+1}\tau L_A}e^{d_n\tau L_{\epsilon B}}\dots e^{d_1\tau L_{\epsilon B}}e^{c_1\tau L_A}e^{-uf_{2n}L_{\mathcal{F}}} \tag{3a}$$

$$CSABA_{2n+1} = e^{-uf_{2n+1}L_{\mathcal{F}}}e^{c_1\tau L_A}e^{d_1\tau L_{\epsilon B}}\dots e^{c_{n+1}\tau L_A}e^{d_{n+1}\tau L_{\epsilon B}}e^{c_{n+1}\tau L_A}\dots e^{d_1\tau L_{\epsilon B}}e^{c_1\tau L_A}e^{-uf_{2n+1}L_{\mathcal{F}}} \tag{3b}$$

$$CSBAB_{2n-1} = e^{-u\theta_{2n-1}L_{\mathcal{F}}}e^{d_1\tau L_{\epsilon B}}e^{c_2\tau L_A}\dots e^{d_n\tau L_{\epsilon B}}e^{c_{n+1}\tau L_A}e^{d_n\tau L_{\epsilon B}}\dots e^{c_2\tau L_A}e^{d_1\tau L_{\epsilon B}}e^{-u\theta_{2n-1}L_{\mathcal{F}}} \tag{3c}$$

$$CSBAB_{2n} = e^{-u\theta_{2n}L_{\mathcal{F}}}e^{d_1\tau L_{\epsilon B}}e^{c_2\tau L_A}\dots e^{c_{n+1}\tau L_A}e^{d_{n+1}\tau L_{\epsilon B}}e^{c_{n+1}\tau L_A}\dots e^{c_2\tau L_A}e^{d_1\tau L_{\epsilon B}}e^{-u\theta_{2n}L_{\mathcal{F}}} \tag{3d}$$

where $u = \frac{1}{2}\tau^3\epsilon^2$. All the above integrators are of the order $O(\tau^\kappa\epsilon + \tau^4\epsilon^2)$, with $\kappa = \nu + 2$, if ν is even and $\kappa = \nu + 3$, if ν is odd. The values of the various constants c_j, d_j, f_j and θ_j can be found in [5].

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USE OF THE CSABA$_\nu$ & CSBAB$_\nu$

The particle dynamics is described by different Hamiltonians composed by the various local magnetic multipoles in the lattice. In this paper, the approximations of the Hamiltonians for high-energy and large machines are used. Their general form is $H = A(p^2, q^n) + B(q^m)$ with $n = 0, 1$ & $m = 0, 1, 2, \ldots$. After calculating the term $\mathcal{F} = \{\{A, B\}, B\}$, it can be shown that it depends only on positions q_i. So $\mathcal{F}$ is integrable and the CSABA$_\nu$ & CSBAB$_\nu$ can be used. Its general form for different multipoles - except for dipoles ($m = 1$), is a non-Maxwellian potential given by:

$$\mathcal{F}(x, y) = \frac{1}{1+\delta}\left(\frac{eB_0}{P_0}\sigma_{m-1}\right)^2 r^{2(m-1)} \quad , \qquad (4)$$

with $\sigma_{m-1} = (b_{m-1}, a_{m-1})$ being the $(2m)^{th}$ multipole coefficients, b_{m-1} refers to the normal and a_{m-1} to the skew multipoles. The B_0 is the main dipole field, P_0 is the reference momentum, e is the electric charge, $r = \sqrt{x^2 + y^2}$ and $\delta = \Delta P/P_0$ the relative momentum deviation.

In earlier works related to accelerator problems [7, 8] the use of the CSABA$_\nu$ & CSBAB$_\nu$ integrators provided very good results. For this paper, they are used in order to integrate the motion of a particle in a quadruple and a FODO cell. Different parameters are calculated and compared with the results taken from the improved TEAPOT integrator [9] that is used in MAD-X [10].

The quadrupole Hamiltonian after some approximations [11] gets the following form:

$$H_Q(x, y, l, p_x, p_y, \delta; s) = \frac{p_x^2 + p_y^2}{2(\delta+1)} + \frac{K1_0}{2}(x^2 - y^2) \quad , \quad (5)$$

with $K1_0$ being the quadrupole's strength ($K1_0 > 0$ for a focusing quadrupole and $K1_0 < 0$ for defocusing one).

The maps of the different Lie transformations that are needed to construct the CSABA$_\nu$ & CSBAB$_\nu$ integrators with $\epsilon \equiv 1$, $A \equiv \frac{p_x^2+p_y^2}{2(\delta+1)}$, $B \equiv \frac{K1_0}{2}(x^2 - y^2)$, $\mathcal{F}_Q = \frac{K1_0^2}{1+\delta}r^2$, $\tau \equiv L_Q$ and $q' = \{H_Q, p_q\} = p_q/(1+\delta)$ are given below:

$$e^{\tau c_j L_A} : \begin{cases} x^f = x^i + c_j L_Q x'^i, & x'^f = x'^i \\ y^f = y^i + c_j L_Q y'^i, & y'^f = y'^i \end{cases} \quad ,$$

$$e^{\tau d_j L_B} : \begin{cases} x^f = x^i, & x'^f = x'^i - d_j \frac{K1_0}{\delta+1} L_Q x^i \\ y^f = y^i, & y'^f = y'^i + d_j \frac{K1_0}{\delta+1} L_Q y^i \end{cases} \quad ,$$

$$e^{u\theta_j L_{\mathcal{F}_Q}} : \begin{cases} x^f = x^i, & x'^f = x'^i + \theta_j (\frac{K1_0}{\delta+1})^2 L_Q^3 x^i \\ y^f = y^i, & y'^f = y'^i + \theta_j (\frac{K1_0}{\delta+1})^2 L_Q^3 y^i \end{cases} \quad .$$

Calculation of the Accuracy Order

The improved TEAPOT is obtained after developing a new splitting pattern. The primarily aim of this was to well describe the $M_{2,1}$ element of the quadrupole's exact matrix. In general, the same splitting for all the other multipoles of the lattice can not be applied. On the other hand, the CSABA$_\nu$ & CSBAB$_\nu$ integrators are completely generic.

Each element $M^q_{j,k}$ of the quadrupole exact matrix M^q, is Taylor expanded and compared with the corresponding $M_{j,k}$ from the CSABA$_\nu$ and TEAPOT$_\nu$ integrators. The order of accuracy ζ for each element of the integrators is plotted in Fig. 1. The numbers in the parentheses refer to the number of kicks and the total maps needed to construct the integrator, e.g. for (4, 7) the integrator consists of 7 maps from which the 4 are kicks and the other drifts. The CSABA$_\nu$ gives equal (for the CSABA$_1$) or better description than TEAPOT$_\nu$ for all the elements $M_{j,k}$. TEAPOT splittings improve slowly with the increase of the number of kicks. The overall behaviour M of the CSABAs is two orders of magnitude better than all the TEAPOTs. The only exception is the CSABA$_1$ which is equivalent to the TEAPOTs behaviour. The CSBABs have the same results with the CSABAs.

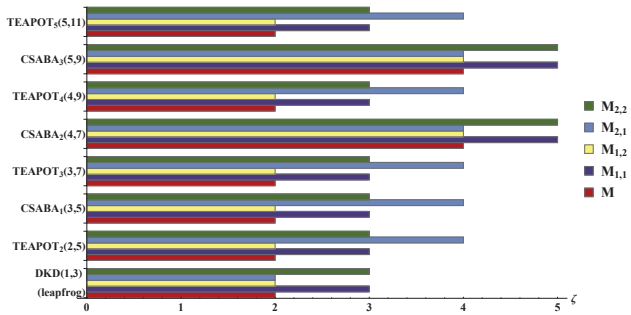

Figure 1: The order of accuracy ζ for each element $M_{j,k}$ using the CSABA$_\nu$ and TEAPOT$_\nu$ integrators.

Calculation of the Phase Advance

Studying a physical quantity can provide a better understanding of the CSABA$_\nu$ & CSBAB$_\nu$ integrators validity. Hence, the quadrupoles' phase advance is calculated using different integrators. For a linear non-periodic lattice, the phase advance μ between the positions s^i and s^f is given by the following equation:

$$Cot(\mu) = \frac{M_{1,1}}{M_{2,1}}\beta^i + \alpha^i \quad , \qquad (6)$$

where β^i and α^i are the Courant-Snyder parameters at s^i and $M_{j,k}$ are the transfer matrix elements from s^i to s^f.

For the different integrators, the absolute value of the relative difference of the ratio $M_{1,1}/M_{2,1}$ with respect to the ratio $M^q_{1,1}/M^q_{2,1}$ of the exact quadrupole is studied. Some characteristic results are presented in Fig. 2. The CSABA$_2$ in Fig. 2b, has a good accuracy (deep purple colour) for a greater area of quadrupole strength $K1_0$ and length L_Q, when compared to TEAPOT$_5$ shown in Fig. 2a. TEAPOT$_5$ consists of a higher number of maps, 11 in contrast to the 7 maps of the CSABA$_2$. Indeed, the CSABA$_\nu$ and similarly the CSBAB$_\nu$ are not only more accurate than TEAPOTs but they

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are also more economical with respect to integration time. In Fig. 2b and 2a, the area between the white dashed lines corresponds to stable motion through a symmetric FODO cell, with the length of each drift being 20m.

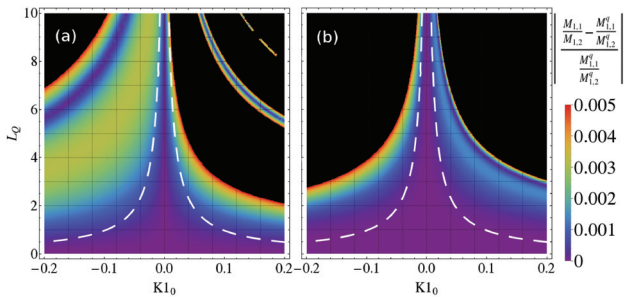

Figure 2: The absolute value of the relative difference of the ratio $M_{1,1}/M_{2,1}$ from the ratio $M^q_{1,1}/M^q_{2,1}$ as a function of different $K1_0$ and L_Q, for the TEAPOT$_5$ (a) and the CSABA$_2$ (b). The area between the white dashed lines guarantees stable motion through a symmetric FODO cell.

Calculation of the Phase Advance Distortion

The phase advance distortion $\Delta\mu$ of an off momentum particle, traveling in a non-symmetric FODO cell (or DOFO if $K1_0 < 0$), is calculated using the CSABA$_\nu$ - CSBAB$_\nu$ and TEAPOT$_\nu$. In an accelerator with periodic structure, that consists of mirror symmetric cells, the phase advance distortion between two mirror symmetric points is calculated by:

$$\Delta\mu = \frac{M_{1,1} - \widetilde{M}_{1,1}}{\sqrt{1 - M^2_{1,1}}} \quad , \tag{7}$$

where $\widetilde{M}$ is the transfer matrix of the perturbed cell and M of the unperturbed one.

Using the different integrators, the absolute value of the relative difference $\Delta\mu$ with respect to $\Delta\mu_{\text{fodofEQ}}$ that is calculated using the quadrupole's exact matrix is studied. Some characteristic results are presented in Fig. 3. In this particular case, $\Delta P/P = 1\%$ and the central quadrupole strength is 20% weaker than the ones at the boundaries of the cell. The results from the CSABA$_3$ can be seen in Fig. 3b and the ones from the TEAPOT$_4$ in Fig. 3a. As can be seen, the two integrators have a more similar behaviour. The CSABA$_3$ results are slightly better for larger values of the $|K1_0|$. Also, the CSABA$_3$ gives symmetric results with respect to the positive/negative values of $K1_0$, but this is not the case for the TEAPOT$_4$.

CONCLUSION

The CSABA$_\nu$ & CSBAB$_\nu$ that are high order symplectic schemes have only positive integration steps. They are used in order to calculate the particle's dynamics in different linear lattices. The results are compared with the improved TEAPOT ones (an integration method that constructed specially for linear elements). Obviously the CSABA$_\nu$ & CSBAB$_\nu$ calculate with similar and most of

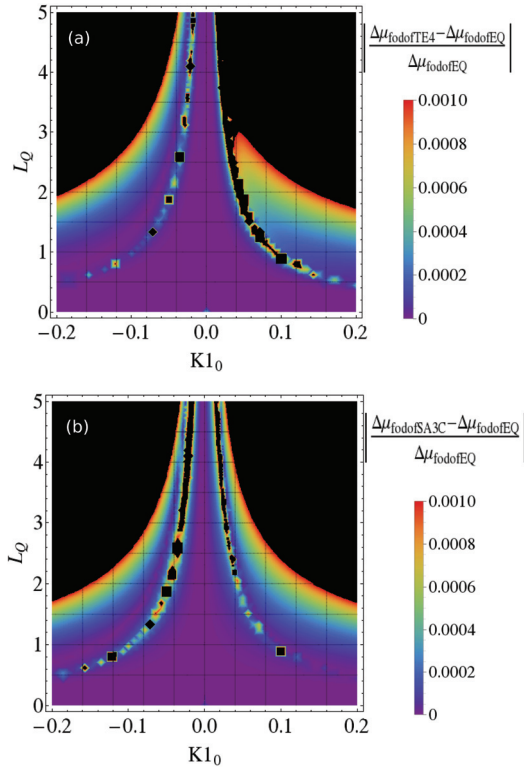

Figure 3: Using the different integrators, the absolute value of the relative difference of $\Delta\mu$ from $\Delta\mu_{fodofEQ}$ is calculated, for the TEAPOT$_4$ (a) and the CSABA$_3$ (b).

the times greater accuracy the parameters studied. In addition, for results having similar accuracy with TEAPOT, the CSABAs and CSBABs need less maps and the integration is faster. Furthermore, the CSABAs & CSBABs can be used to integrate the particle's motion though a nonlinear lattice, a study that is currently under development.

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